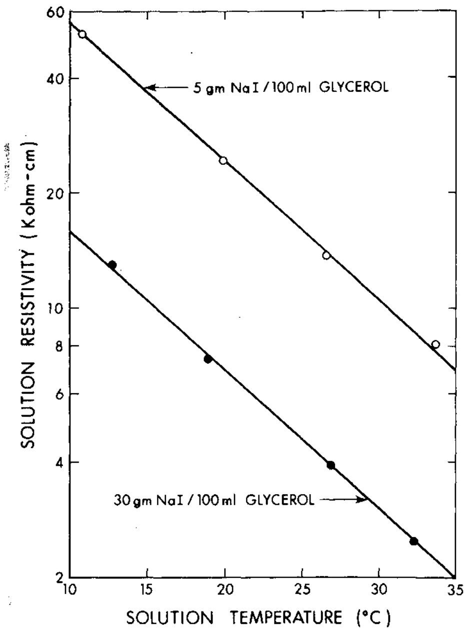

Fig. 2 Solution resistivity at 1 kHz vs temperature for outgassed 5 g and 30 g NaI/100 ml glycerol solutions.

Solutions were outgassed by liquid nitrogen trapped pumping on about 50 ml of solution in a 250 ml Erlenmeyer flask immersed in a 65°C bath. The resistivity of a 30 g NaI/100 ml glycerol solution rose from 2.6 kohm-cm initially to 4.5 kohm-cm after two hours of outgassing, and rose only 15% more in a further sixteen hours of outgassing. The characteristic yellow color of the NaI-glycerol solutions disappeared after about four hours of outgassing. A standard time of five hours of outgassing at 65°C was used for solution preparation.

The resistivities of various NaI-glycerol solutions are shown in Table 2. Two different glycerol grades were tested at doping levels of 12 g and 30 g/100 ml glycerol. The resistivity of USP grade glycerol solutions was initially lower than that of the reagent grade glycerol solutions, but outgassing essentially removed the difference. A total of six doping levels were measured ranging from pure glycerol to 30 g NaI/100 ml glycerol, in both original and outgassed form. It was found that solutions with a particular resistivity value could be reproduced within ±5%.

The measured resistivity values show general agreement with the values from the literature (Table 1). The values of Makin and Bright[5] are inexplicably high.

Some recommendations on solution resistivity measurement procedure can be made from the observed dependence of resistivity on the experimental conditions. The frequency dependence of cell impedance must be examined to find a range in which the impedance is independent of frequency, thus allowing an unambiguous determination of resistivity. The strong temperature dependence of resistivity makes temperature control essential. This can be done by using a controlled environment or applying an empirical temperature normalization, but the cell must be allowed to completely stabilize before measurement in either case. Solutions should be outgassed before measurement. Once outgassed, solution resistivity depends only on doping level, and is stable under vacuum. If these precautions are observed the resistivities of NaI-glycerol solutions are clearly defined, stable, and reproducible.

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Use of Polynomial Approximations to Calculate Suboptimal Controls

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Introduction

MANY methods have been developed to calculate optimal and suboptimal control laws for control problems.[1] The purpose of this Note is to present a new method for calculating suboptimal controls. The method is based on assuming a functional form such as a polynomial for the control. The functional form will contain several arbitrary constants. These constants are then selected to produce a control which causes the trajectory to meet terminal conditions and extremize the performance index in some sense.

Problem Statement and Method of Solution

The problem to be considered may be stated in the following manner: minimize

$$I = G(\mathbf{x}_f, t_f) \tag{1}$$

subject to

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \tag{2}$$

and

$$\mathbf{x}_o = \mathbf{x}_{os} \qquad \mathbf{M}(\mathbf{x}_f, t_f) = 0 \tag{3}$$

where $\mathbf{x}$ is an n vector of state variables, $\mathbf{u}$ is an m vector of control variables, t is time, $\mathbf{x}_{os}$ is the specified initial state vector, G is a scalar performance index, and $\mathbf{M}$ is a p vector of terminal conditions.

The control $\mathbf{u}$ is now chosen as some arbitrary function, such as a polynomial in time, with arbitrary constant coefficients. Thus

$$\mathbf{u} = U(\mathbf{a}, t) \tag{4}$$

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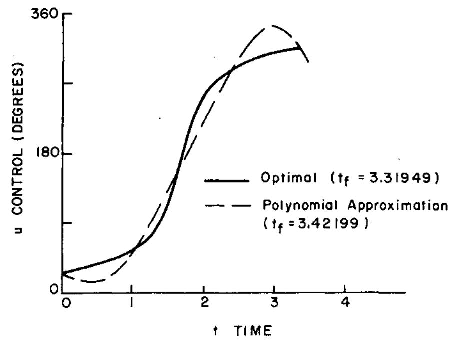

Fig. 1 Control vs time for low thrust example.

where U is the assumed function and $\mathbf{a}$ is a g vector of constant coefficients. In general, g will be chosen to be greater than p. If the problem is now linearized then

$$\delta\dot{\mathbf{x}} = \mathbf{f}_x\delta\mathbf{x} + \mathbf{f}_u\delta\mathbf{u}, \delta\mathbf{u} = U_a\delta\mathbf{a} \tag{5}$$

so that

$$\delta\dot{\mathbf{x}} = \mathbf{f}_x\delta\mathbf{x} + \mathbf{f}_u U_a\delta\mathbf{a} \tag{6}$$

If the terminal conditions are linearized then

$$\Delta\mathbf{M} = \mathbf{M}_{x_f}\delta\mathbf{x}_f + \dot{\mathbf{M}}\Delta t_f \tag{7}$$

The state at the final time can then be linearly related to changes in $\mathbf{a}$ by

$$\delta\mathbf{x}_f = \boldsymbol{\phi}(f,0)\delta\mathbf{a} \tag{8}$$

where ϕ is an $\mathbf{n} \times \mathbf{g}$ matrix. Each column is a solution of Eq. (6) corresponding to a unit perturbation in each of the $\mathbf{a}$'s. Boundary conditions for the ϕ matrix also require $\delta\mathbf{x}_o = 0$. Then Eq. (7) may be written as

$$\Delta\mathbf{M} = \mathbf{M}_{x_f}\boldsymbol{\phi}(f,0)\delta\mathbf{a} + \dot{\mathbf{M}}\Delta t_f \tag{9}$$

Several procedures are now possible in order to calculate $\delta\mathbf{a}$ and Δt_f to drive $\mathbf{M}$ to zero and minimize G in some sense. The one used here defines a new performance index for the linear problem to be

$$I^* = w[G(\mathbf{x}_f,t_f) + G_{x_f}\boldsymbol{\phi}(f,0)\delta\mathbf{a} + G_{t_f}\Delta t_f] + \tfrac{1}{2}\delta\mathbf{a}^T\delta\mathbf{a} \tag{10}$$

where w is a scalar weighing constant. The term in brackets is a linear approximation to G for the $(i + 1)$th trajectory. As $\Delta\mathbf{M}$ goes to zero, $\delta\mathbf{a}$ and Δt_f will hopefully go toward zero from Eq. (9). Then Eq. (10) becomes essentially I. If $\delta\mathbf{a}$ and Δt_f do become small, then minimizing I^* should produce a reasonable approximation of the true optimal. To minimize Eq. (10) subject to Eq. (9), form the augmented performance index

$$I = I^* + \boldsymbol{\lambda}^T[\mathbf{M}_{x_f}\boldsymbol{\phi}(f,0)\delta\mathbf{a} + \dot{\mathbf{M}}\Delta t_f - \Delta\mathbf{M}] \tag{11}$$

Then require $\partial I/\partial\delta a = 0$ and $\partial I/\partial\Delta t_f = 0$. These conditions require that

$$\delta\mathbf{a} = -wG_{x_f}\phi - \boldsymbol{\phi}^T\mathbf{M}_{x_f}{}^T\boldsymbol{\lambda}, \quad \Delta t_f = -wG_{t_f} - \dot{\mathbf{M}}^T\boldsymbol{\lambda} \tag{12}$$

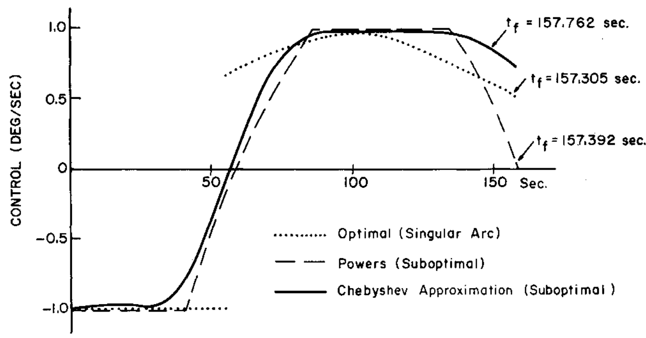

Fig. 2 Control vs time for singular arc problem.

Table 1 Nominal and converged values for the a vector for the low thrust example

Variable	Nominal	Converged
a_o	0.3	0.4352902546
a_1	5.0	− 3.552988052
a_2	0.0	18.32013697
a_3	0.0	5.47435134
a_4	0.0	−15.53131576
$a_5(t_f)$	3.4	3.424558894

where

$$\boldsymbol{\lambda} = -(\mathbf{M}_{x_f}\boldsymbol{\phi\phi}^T\mathbf{M}_{x_f}{}^T + \dot{\mathbf{M}}\dot{\mathbf{M}}^T)^{-1}(\Delta\mathbf{M} + w\mathbf{M}_{x_f}\boldsymbol{\phi\phi}^T G_{x_f}{}^T + w\dot{\mathbf{M}}G_{t_f}) \tag{13}$$

Thus the numerical procedure involves guessing the $\mathbf{a}$ vector and t_f. The state equations and the $\boldsymbol{\phi}$ matrix are integrated from the initial time to t_f. Eqs. (12) and (13) are used to solve for $\delta\mathbf{a}$ and Δt_f. These corrections are added to the guessed variables to produce new values for $\mathbf{a}$ and t_f. The state equations and $\boldsymbol{\phi}$ matrix are reintegrated and new corrections are calculated. This procedure continues until $\mathbf{M}$ is near zero and G is as small as desired.

Note that in general, $\delta\mathbf{a}$ and Δt_f are scaled to keep the linear approximations involved in calculating these values from being violated. Also, if a polynomial in time is assumed, then time, the independent variable should probably be normalized between zero and one. This is done by defining $t = \mathbf{a}_{g+1}\Upsilon$ where $0 \leq \Upsilon \leq 1$ and $\mathbf{a}_{g+1}$ is another unknown constant which must be determined by the iteration procedure. Both the normilization of time and scaling of the corrections are performed for the numerical examples.

Examples

The first example considered is a low thrust Earth to Mars transfer problem.[2] Time is to be minimized so

$$G(\mathbf{x}_f,t_f) = t_f \tag{14}$$

The state equations are

$$\begin{aligned} \dot{u} &= v^2/r - \mu/r^2 + T\sin\beta/m \\ \dot{v} &= -uv/r + T\cos\beta/m \\ \dot{r} &= u \end{aligned} \tag{15}$$

where u is the radial velocity, v is the tangential velocity, r is the radial distance from the sun to the spacecraft, T is the thrust magnitude, m is the mass of the vehicle, μ is the gravitational constant, and β is the control. The mass flow is linear so that $m = m_o - \dot{m}t$ where m_o and $\dot{m}$ are constants. In normalized units, $\mu = 1.0$, $m_o = 1.0$, $\dot{m} = 0.074800391$, and $T = 0.14012969$. Boundary conditions require $u_o = 0.0$, $v_o = 1.0$, and $r_o = 1.0$. Also

$$\mathbf{M}^T(\mathbf{x}_f,t_f) = (u_f, v_f - 0.81012728, r_f - 1.523679) \tag{16}$$

For this example, a fourth order polynomial in the normalized independent variable, Υ, is assumed for β. Thus

$$\beta = \mathbf{a}_o + \mathbf{a}_1\Upsilon + \mathbf{a}_2\Upsilon^2 + \mathbf{a}_3\Upsilon^3 + \mathbf{a}_4\Upsilon^4 \tag{17}$$

Table 2 Nominal and converged values for the a vector for the Saturn singular arc problem

Variable	Nominal	Converged
a_o	0.0	0.1323139
a_1	1.7	1.7493868
a_2	0.0	−0.64295498
a_3	0.0	−0.56766565
a_4	0.0	0.17679134
$a_5(t_f)$	0.25	0.19577421

Initial guesses for the **a** vector are shown in Table 1. The converged values are also shown in Table 1. The optimal control, shown in Ref. 2, and the control obtained here are shown in Fig. 1.

Approximately 25 integrations of the ϕ matrix were required to produce this trajectory. The iterations were stopped since this trajectory has $||M|| \leq 10^{-5}$ and although G is still decreasing on every iteration, it is decreasing very slowly.

The second example considered is a singular arc problem solved by Powers.[3] Again the quantity to be minimized is the final time. The state equations are the same as those shown in Eqs. (15) except that one more equation is added. The control of the Saturn vehicle is the rate of change of β. Thus if the following equation is introduced

$$\dot{\beta} = \gamma \tag{18}$$

then β becomes a state variable and the control is γ. This choice of control causes the state equations to be linear in the control variable.

For the numerical computations, variables are normalized as before where the unit of length is now chosen as one earth radius. Boundary conditions then require $u_o = 0.0220159627$, $v_o = 0.853457258$, $r_o = 1.03066079$, and $\beta_o = 0.2987038$. At the final time $u_f = 0.0$, $v_f = 0.9853923$, and $r_f = 1.02987038$. It is also required that $-K \leq \gamma \leq K$. This constraint on the control is easily satisfied by assuming the control to be of the form

$$\gamma = K \sin U(\mathbf{a},\Upsilon) \tag{19}$$

where U is a polynomial.

Initially U was specified to be a fourth order polynomial just as before. After a few iterations were made, however, it was obvious that this would not be sufficient to produce converged trajectories. The rows of the ϕ matrix were essentially linearly dependent. This resulted in very poor accuracy for the corrections in the **a** vector. This problem was easily solved by using Chebyshev polynomials normalized over the interval zero to one instead of a standard polynomial. The use of Chebyshev polynomials kept the rows from becoming linearly dependent.

For this case, initial and converged values for **a** are shown in Table 2. The control calculated here is compared with the control programs calculated by Powers in Fig. 2. Only 9 iterations were required to calculate the trajectory.

Due to the normalization, w was always set equal to one. By increasing w it is possible to produce trajectories with a slightly lower performance index than the one shown. It is also expected that a higher order polynomial could be used to produce a trajectory with a smaller performance index. The purpose of the note, however, is to show that it is relatively easy to guess a polynomial which can be used to produce reasonably good results. Thus the questions concerning the choice of best order for the polynomial and best weighting for convergence are not discussed.

Results

The possibility of guessing a functional form for a control variable is shown to be a reasonable method for calculating good suboptimal control laws. By guessing a functional form, it is necessary to calculate corrections to only a few variables rather than the entire control history. This makes the numerical algorithm used to solve optimal control problems very simple.

The results for the singular arc problem are particularly encouraging. The method used is a first order method and it converges rapidly for this problem. By experimenting with the order of the polynomial chosen and selecting the proper weighting constant, it should be possible to produce very good suboptimal control laws for this problem.

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Vibration of Slightly Curved Beams of Transversely Isotropic Composite Materials

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Material systems such as fiber reinforced plastic composites and pyrolytic graphite type materials are finding increased use in structural applications. Many of these materials are transversely isotropic,[1,2] for which the ratio of in-plane modulus of elasticity to shear modulus E/G is large, with values ranging from 20 to 50. For such values of E/G, results of Ref. 1 indicate that even in relatively thin beams and plates, the effects of transverse shear deformation are important in reducing the natural frequency of flexural vibration. In the present Note, slightly curved transversely isotropic beams are analyzed, and simple approximate analytical results are derived which indicate how even very slight curvature tends to increase sharply the natural frequency.

The fundamental equations to be derived and used are those of a slightly curved Timoshenko beam. For beams of small curvature, the equations of motion can be taken as follows, with the notation of Fig. 1:

$$M' + V = \mu\ddot{\phi} \tag{1a}$$

$$N' = \rho\ddot{u} \tag{1b}$$

$$V' + (w_0'N)' = \rho\ddot{w} \tag{1c}$$

Similar equations were used in Ref. 3 but without the $\mu\ddot{\phi}$ term. Here, $w_0(x)$ is the initial shape of the beam axis, $\phi(x)$ is the rotation of cross sections, and $w'(x)$ is the rotation of tangents to the beam axis. Also, $\mu = \bar{\gamma}I$ where $\bar{\gamma}$ is a volume density; ρ is a linear density so that $\rho = \bar{\gamma}A$. The beam cross-sectional area and moment of inertia are given by A and I.

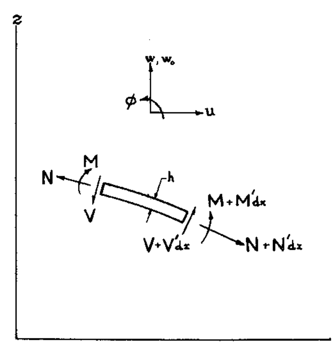

Fig. 1 Stress resultants N,V, moment M, rotation ϕ, and displacements w,u, on beam element.

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